EXPONENTIAL BOUNDS IN THE LAW OF ITERATED LOGARITHM FOR MARTINGALES

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Abstract. In this paper non-asymptotic exponential estimates are derived for tail of maximum martingale distribution by naturally norming in the spirit of the classical Law of Iterated Logarithm.
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1. Introduction. Notations. Statement of problem.

Let $(\Omega, F, \mathbf{P})$ be a probability space, $\Omega = \{\omega\}, \ (S(n), F(n)), \ n = 1, 2, \ldots$ being a centered: $\mathbf{E}S(n) = 0$ non-trivial:

$$\forall n \ \Rightarrow \sigma(n) = [\mathbf{Var} \ (S(n))]^{1/2} \in (0, \infty)$$

martingale: $\mathbf{E}S(n+1)/F(n) = S(n)$ relatively some filtration $F(n)$. Let also $v(n)$ be a *deterministic* positive monotonically increasing sequence, $A(k)$ be a deterministic positive strong monotonically increasing *integer* sequence $A(k), \ k = 1, 2, \ldots$ such that $A(1) = 1, \ B(k) \stackrel{def}{=} A(k+1) - 1 \ge A(k) + 1$. Introduce the partition of integer semi-axis $Z_+ = [1, 2, \ldots) \ R = \{A(k), B(k)\}:$

$$Z_+ = \cup_{k=1}^{\infty} [A(k), B(k)] = \cup_{k=1}^{\infty} [A(k), A(k+1) - 1]$$

and denote the *set* of all these partitions by $T: \ T = \{R\}$.

Let us introduce the following probability $W(u)$:

$$W(u) = W(v;u) \stackrel{def}{=} \mathbf{P}\left(\sup_n \frac{S(n)}{\sigma(n)\ v(n)} > u\right), \tag{1}$$

and analogously set

$$W_+(u) = W_+(v;u) \stackrel{def}{=} \mathbf{P}\left(\sup_n \frac{|S(n)|}{\sigma(n)\ v(n)} > u\right).$$

Our goal is obtaining the exponential decreasing estimation for $W(u),\ W_+(v,u)$ for sufficiently greatest values u, for example, $u \ge 2$.

In the case when $S(n) = \sum_{i=1}^n \xi(i)$, where $\{\xi(i)\}$ are independent centered r.v. and $\sigma -$ flow $\{F(n)\}$ is the natural filtration:

$$F(n) = \sigma\{\xi(i),\ i = 1, 2, \ldots, n\}$$

with the classical norming $v(n) = (\log(\log(n+3))^{1/2}$ the estimation for $P(u)$ was obtained in [1], see also [2], p.62 - 66. Our result may be considered as some addition to the classical Law of Iterated Logarithm (LIL) for martingales, i.e. of the view

$$\overline{\lim}_{n\to\infty} |S(n)|/(\sigma(n)\ v(n)) = \eta(\omega) < \infty \ \mathbf{a.e.}, \tag{2}$$

see [3], p.115-127 and references there.

It is clear that if the conclusion (2) is satisfied, then the bound for $P(u)$ is not trivial, i.e. $u \to \infty \ \Rightarrow P(u) \to 0$.

2. Result.

In order to formulate our result, we need to introduce some another notations and conditions. Let $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0),\ \lambda_0 = const \in (0, \infty]$ be some even taking positive values for positive arguments strong convex twice continuous differentiable function, such that

$$\phi(0) = 0,\ \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty. \tag{3}$$

The set of all these function we denote Φ; $\Phi = \{\phi(\cdot)\}$. We say that the *centered* random variable (r.v) $\xi = \xi(\omega)$ belongs to the space $B(\varphi)$, if there exists some non-negative constant $\tau \ge 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \ \Rightarrow \mathbf{E} \exp(\lambda \xi) \le \phi(\lambda \, \tau). \qquad (4).$$

The minimal value τ satisfying (4) is called the $B(\phi)$ norm of the variable ξ, write

$$||\xi|| B(\phi) = \inf\{\tau, \ \tau > 0 : \ \forall \lambda \ \Rightarrow \mathbf{E} \exp(\lambda \xi) \le \exp(\phi(\lambda \, \tau))\}.$$

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of limit theorem, exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields etc.

The space $B(\phi)$ relative to the norm $|| \cdot || B(\phi)$ is a Banach space which is isomorphic to subspace consisted on all the centered variables of Orlichs space $(\Omega, F, \mathbf{P}), N(\cdot)$ with $N -$ function

$$N(u) = \exp(\phi^*(u)) - 1, \ \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \to \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [2, p. 19 - 40], [4]:

$$\mathbf{1}. \ \xi \in B(\phi) \Leftrightarrow \mathbf{E}\xi = 0, \ \textbf{and} \ \exists C = const > 0,$$

$$U(\xi, x) \le \exp(-\phi^*(Cx)), x \ge 0,$$

where $U(\xi, x)$ denotes as usually the tail of distribution of a r.v. ξ :

$$U(\xi, x) = \max\left(\mathbf{P}(\xi > x), \ \mathbf{P}(\xi < -x)\right), \ x \ge 0, \qquad (5)$$

and this estimation (5) is in general case asymptotically exact.

Here and further $C, C_j, C(i)$ will denote the non-essentially positive finite "constructive" constants.

More exactly, if $\lambda_0 = \infty$, then the following implication holds:

$$\lim_{\lambda \to \infty} \phi^{-1}(\log \mathbf{E} \exp(\lambda \xi))/\lambda = K \in (0, \infty)$$

if and only if

$$\lim_{x \to \infty} (\phi^*)^{-1}(|\log U(\xi, x)|)/x = 1/K.$$

Here and further $f^{-1}(\cdot)$ denotes the inverse function to the function f on the left-side half-line (C, ∞).

2. Define $\psi(p) = p/\phi^{-1}(p), \ p \ge 2$. Let us introduce the new norm on the set of r.v. defined in our probability space by the following way: the space $G(\psi)$ consist, by definition, on all the centered r.v. with finite norm

$$||\xi||G(\psi) \stackrel{def}{=} \sup_{p \ge 2} |\xi|_p/\psi(p), \ |\xi|_p = \mathbf{E}^{1/p}|\xi|^p. \tag{6}$$

It is proved that the spaces $B(\phi)$ and $G(\psi)$ coincides:$B(\phi) = G(\psi)$ (set equality) and both the norm $|| \cdot ||B(\phi)$ and $|| \cdot ||$ are equivalent: $\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = const \in (0, \infty), \ \forall \xi \in B(\phi)$

$$||\xi||G(\psi) \le C_1 \ ||\xi||B(\phi) \le C_2 \ ||\xi||G(\psi).$$

3. The definition (6) is correct still for the non-centered random variables ξ. If for some non-zero r.v. ξ we have $||\xi||G(\psi) < \infty$, then for all positive values u

$$\mathbf{P}(|\xi| > u) \le 2 \ \exp\left(-u/(C_3 \ ||\xi||G(\psi))\right). \tag{7}$$

and conversely if a r.v. ξ satisfies (7), then $||\xi||G(\psi) < \infty$.

We suppose in this article that there exists the function $\phi \in \Phi$ such that

$$\sup_n [||S(n)||B(\phi)/\sigma(n)] < \infty,$$

or equally for all non-negative values x

$$\sup_n \max \left[\mathbf{P}\left(\frac{S(n)}{\sigma(n)} > x \right), \ \mathbf{P}\left(\frac{S(n)}{\sigma(n)} < -x \right) \right] \le \exp\left(-\phi^*(x/C)\right). \tag{8}$$

The function $\phi(\cdot)$ may be constructive introduced by the formula

$$\phi(\lambda) = \log \sup_n \mathbf{E} \exp(\lambda S(n)/\sigma(n)),$$

if obviously the family of r.v. $\{S(n)/\sigma(n)\}$ satisfies the *uniform* Kramer's condition: $\exists \mu \in (0, \infty), \ \forall x > 0 \ \Rightarrow$

$$\sup_n U(S(n)/\sigma(n), \ x) \le \exp(-\mu \ x).$$

There are many examples of martingales satisfying the condition (8) in the article [5]; in particular, there are many examples with

$$\phi^*(x) = x^r \ L(x), \ r = const > 0, \tag{9}$$

$$n^\gamma \ M_1(n) \le \sigma(n) \le n^\gamma \ M_2(n), \ \gamma = const > 0, \tag{10}$$

where $L(x), M_1(n), M_2(n)$ are some positive continuous *slowly varying* as $x \to \infty$ or correspondently as $n \to \infty$ functions.

Let us denote for some partition $R = \{A(k), B(k)\}$

$$Q(k; R, v, u) = \exp\left(-\phi^*(u\sigma(A(k)) \ v(A(k))/\sigma(B(k))\right),$$

$$Q(R, v, u) = \sum_{k=1}^{\infty} Q(k; R, v, u). \tag{11}$$

Theorem. *Under our conditions and for some finite* $C = C(\phi)$

$$W(v; u) \le \inf_{R \in T} Q(R, v, Cu), \tag{12}$$

and analogous estimation is true for the probability $W_+(v, u)$.

Proof. Let $Z_+ = \cup_k [A(k), B(k)], \ B(k) = A(k+1) - 1$ be arbitrary partition, $R = \{A(k), B(k)\} \in T$. Denote $E(k) = [A(k), B(k)]$. We see:

$$W(v; u) \le \sum_{k=1}^{\infty} W(k; v, u), \ W(k; v, u) \stackrel{def}{=}$$

$$\mathbf{P}\left(\max_{n \in E(k)} (S(n)/(\sigma(n) \ v(n)) > u\right). \tag{13}$$

Let us estimate the probability $W(k; v, u)$. We obtain:

$$W(k;v,u) \le \mathbf{P}\left(\max_{n \in E(k)} S(n) > u\ \sigma(A(k))\ v(A(k))/\sigma(B(k))\right),$$

as long as both the functions $\sigma(\cdot)$ and $v(\cdot)$ are monotonically increasing.

Further we use the Doob's inequality and properties of $B(\phi)$ spaces. It follows from Doob's inequality

$$|\max_{n \in E(k)} S_n|_p \le C\ \sigma(B(k)) \cdot (p/\phi^{-1}(p)) \cdot (p/(p-1)) \le$$

$$2\ C\ \sigma(B(k)) \cdot (p/\phi^{-1}(p))$$

as long as $p \ge 2$. Therefore $W(k;v,u) \le$

$$\exp\left(-\phi^*(Cu\ \sigma(A(k))\ v(A(k))/\sigma(B(k))\right) = Q(k;R,v,Cu).$$

We obtain alter summation

$$W(v;u) \le Q(R,v,Cu).$$

Since the partition R is arbitrary, we get to the demanded inequality (12).

The probability $W_+(v;u)$ is estimated analogously, as long as $(-S(n), F(n))$ is again the martingale with at the same function $\phi(\cdot)$.

Note that we can ground our theorem from the Kolmogorovs inequality for martingales.

3. Examples. Let us consider some examples in order to show the exactness of our theorem.

A. Let η be a symmetrically distributed r.v. with the tail of distribution of a view:

$$\mathbf{P}(\eta > x) = \exp\left(-\phi^*(x)\right),$$

$x \ge 0,\ \phi \in \Phi$; and let $\{\xi(i)\}$ be an independent copies of η. Then $||\eta||B(\phi) = C_5 \in (0,\infty),\ \beta^2 \stackrel{def}{=} \mathbf{Var}(\eta) \in (0,\infty)$.

Let us consider the martingale $(S(n), F(n))$, where

$$S(n) = \sum_{k=1}^{n} 2^{-k}\ \xi(k)$$

relative the natural filtration $\{F(n)\}$. It follows from the triangle inequality for the $B(\phi)$ norm that

$$\sup_n ||S(n)||B(\phi) \le \sum_{k=1}^{\infty} 2^{-k} \ ||\xi(k)||B(\phi) = C_5 < \infty,$$

$$0.25 \ \beta^2 \le \sigma^2(n) \le \beta^2;$$

therefore

$$\exp\left(-\phi^*(C_6 x)\right) \le \sup_n \mathbf{P}(S(n)/\sigma(n) > x) \le$$

$$\exp\left(-\phi^*(C_7 x)\right),$$

$0 < C_7 < C_6 < \infty$ (the low bound is trivial).

Moreover, it is possible to prove that

$$\inf_n \mathbf{P}(S(n) > x) \ge \exp\left(-\phi^*(C_8 x)\right).$$

B. Assume here that the martingale $(S(n), F(n))$ satisfies the conditions (9) and (10). Let us choose

$$v(n) = v_r(n) = [\log(\log(n+3))]^{1/r},$$

or equally

$$v(n) = v_r(n) = [\log(\log(\sigma(n)+3))]^{1/r},$$

then we obtain after some calculation on the basis of our theorem, choosing the partition $R = \{[A(k), A(k+1)-1]\}$ such that:

$$A(k) = Q^{k-1},$$

where $Q = 3$ or $Q = 4$ etc.:

$$\mathbf{P}\left(\sup_n \frac{S(n)}{\sigma(n) \ v_r(n)} > x\right) \le \exp\left[-C \ x^r \ L(x)\right], x > 0. \qquad (14)$$

Moreover, if the martingale $(S(n), F(n)$ satisfies the conditions (8), (9) and (10), then with probability one

$$\overline{\lim}_{n\to\infty} \frac{S(n)}{\sigma(n) \ v_r(n)} \le C,$$

where the constant C is defined in (8); and the last inequality is exact, e.g., for the martingales considered in the next section **C.**

C. Let us show the exactness of the estimation (14). Consider the so-called Rademacher sequence $\{\epsilon(i)\}, \ i = 1, 2, \ldots$; i.e. where $\{\epsilon(i)\}$ are independent and $\mathbf{P}(\epsilon(i) = 1) = \mathbf{P}(\epsilon(i) = -1) = 0.5$.

It is known that that the r. v. $\{\epsilon(i)\}$ belongs to the $B(\phi_2)$ space with corresponding function

$$\phi_2(\lambda) = 0.5 \ \lambda^2, \ \lambda \in (-\infty, \infty).$$

Denote for $d = 1, 2, 3, \ldots \ S(n) = S_d(n) =$

$$\sum \sum \ldots \sum_{1 \le i(1) < i(2) \ldots < i(d) \le n} \epsilon(i(1)) \ \epsilon(i(2)) \ \epsilon(i(3)) \ldots \ \epsilon(i(d))$$

under natural filtration $F(n)$. It is easy to verify that $(S(n), F(n))$ is a martingale and that

$$0 < C_1 \le \sigma^2(n)/n^d \le C_2 < \infty.$$

It follows from our theorem that

$$\mathbf{P}\left(\sup_n \frac{S(n)}{(n \ \log(\log(n+3)))^{d/2}} > u\right) < \exp\left(-C u^{2/d}\right),$$

and as it is proved in [5]

$$\exp\left[-C_3 \ x^{2/d}\right] \le$$

$$\sup_n \mathbf{P}\left(\frac{|S(n|}{\sigma(n)} > x\right) \le \exp\left[-C_4 \ x^{2/d}\right], x > 0,$$

i.e. in the considered case $r = 2/d$.

We prove in addition that

$$\mathbf{P}\left(\overline{\lim}_{n \to \infty} \frac{S(n)}{(n \ \log(\log(n+3)))^{d/2}} > 0\right) > 0. \tag{15}$$

Ii is enough to consider only the case $d = 2$, i.e. when

$$S(n) = \sum \sum_{1 \le i < j \le n} \epsilon(i) \ \epsilon(j).$$

We observe that

$$2\ S(n) = \left(\sum_{k=1}^{n} \epsilon(k) \right)^2 - \sum_{m=1}^{n} (\epsilon(m))^2 \stackrel{def}{=} \Sigma_1(n) - \Sigma_2(n).$$

From the classical LIL on the form belonging to Hartman-Wintner it follows that there exist a finite non-trivial non-negative random variables $\theta_1,\ \theta_2$ for which

$$|\Sigma_2(n)| \le n + \theta_2 \sqrt{n\ \log(\log(n+3))} \tag{16}$$

and

$$\Sigma_1(n_m) \ge \theta_1\ n_m\ \log(\log(n_m+3)) \tag{17}$$

for some (random) integer positive subsequence $n_m,\ n_m \to \infty$ as $m \to \infty$.

The proposition (15) it follows immediately from (16) and (17).

More exactly, by means of considered method may be proved the following relation:

$$\overline{\lim}_{n\to\infty} \frac{S(n)}{(n\ \log(\log(n+3)))^{d/2}} \stackrel{a.e}{=} \frac{2^{d/2}}{d!}.$$

4. It is easy to prove the non-improvement of the estimation (14). Namely, let us consider the martingale $(S(n), F(n))$ satisfying the conditions (9) and (10) and such that for some $n_0 = 1, 2, 3, \ldots$

$$\mathbf{P}\left(\frac{S(n_0)}{\sigma(n_0)} > u \right) \ge \exp\left(-C_9\ u^r\ L(u)\right);$$

then

$$W(v_r; u) \ge \mathbf{P}\left(\frac{S(n_0)}{\sigma(n_0)\ v_r(n_0)} > u \right) =$$

$$\mathbf{P}\left(\frac{S(n_0)}{\sigma(n_0)} > u\ v_r(n_0) \right) \ge \exp\left(-C_{10}\ u^r\ L(u)\right),$$

since the function $L(\cdot)$ is slowly varying.

4. Concluding remarks.

1. It is evident that only the case when

$$\lim_{n\to\infty} \sigma(n) = \infty$$

is interest.

2. Instead the norm $\sigma(n) = |S(n)|_2$ we can consider some another rearrangement invariant norm in our probability space, say, the L_s norm

$$\sigma_s(n) = |S(n)|_s, s = const \ge 1$$

or some norm in Orliczs space, $B(\nu), \ \nu \in \Phi$ norm etc.

But the norm $\sigma(n)$ is classical and more convenient. For instance, if $S(0) \stackrel{def}{=} 0$, then

$$\sigma^2(n) = \sum_{k=0}^{n-1} \mathbf{Var}(S(k+1) - S(k)).$$

3. The exponential bounds for tail of distribution in the LIL for martingales used, for instance, in the non-parametric statistic by adaptive estimations (see [6]).

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